Analytic Torsion and R-Torsion for Manifolds with Boundary

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Abstract

We prove a formula relating the analytic torsion and Reidemeister torsion on manifolds with boundary in the general case when the metric is not necessarily a product near the boundary. The product case has been established by W. Lück and S. M. Vishik. We find that the extra term that comes in here in the nonproduct case is the transgression of the Euler class in the even dimensional case and a slightly more mysterious term involving the second fundamental form of the boundary and the curvature tensor of the manifold in the odd dimensional case.

1   Introduction

The Reidemeister torsion (R-torsion for short) is a combinatorial/topological invariant associated to a unitary representation of the fundamental group of a manifold. Introduced by Reidemeister [R] and Franz [F], it is used to classify the lens spaces. It was further developed by Milnor and Whitehead and used successfully in classifying the cobordisms. In searching for an analytic interpretation of the R-torsion, Ray and Singer [RS1], [RS2] introduced the analytic torsion, which subsequently found many significant and far-reaching applications (Cf., for example, [B], [BGS], [BL], [GS], [DO], [W]).

The celebrated Cheeger-Müller theorem [C], [Mü1] establishes the Ray-Singer conjecture: namely, on closed manifolds, the analytic torsion is indeed the same as the R-torsion. Further significant work along this line includes that of Müller [Mü2] where he extends the theorem to unimodular representations, that of Bismut-Zhang [BZ], which treats general representations (in which interesting secondary invariants come

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in), and that of Burghelea-Friedlander-Kappeler-McDonald [BFKM] which deals with infinite dimensional representations (finite type Hilbert module).

All these work deals with closed manifolds. With appropriate boundary conditions, the torsion invariants can also be defined for manifolds with boundary. In fact, the torsions for manifolds with boundary provides an important stepping stone in Cheeger's approach to the Cheeger-Müller theorem (Cf. [C], [RS1]). In [C], the analog of Ray-Singer conjecture on manifolds with boundary is briefly discussed but the geometric information involved is not made explicit. This problem is addressed much later, first in Lott-Rothenberg [LR], and then Lück [L] and Vishik [V]. By assuming a product metric structure near the boundary, they found that the analytic torsion and the R-torsion differ by a topological invariant, namely, (up to a constant), the Euler number of the boundary. This work has been generalized by Burghelea-Friedlander-Kappeler [BFK] to the finite type Hilbert module.

We obtain a general formula of the difference of analyitc torsion and Reidemaster torsion on a Riemannian manifold with boundary, without assuming the product metric structure near the boundary. In this case, there are two extra, geometric terms coming in, involving the second fundamental form of the boundary. In fact, one of the terms is the Chern-Simons class defined by Bismut-Zhang [BZ], transgressing the Euler class. This term therefore vanishes if the dimension of the manifold is odd. The other term, which vanishes for the even dimension, is only slightly more complicated, involving the second fundamental form of the boundary and the curvature tensor of the manifold.

To be more precise, let M be a compact Riemannian manifold with boundary ∂M and metric $g = g^{TM}$. If we denote x the geodesic distance to the boundary ∂M, then, near the boundary,

$$g = dx^2 + g_{\partial M}(x),$$

where $g_{\partial M}(x)$ is a family of Riemannian metrics on the boundary. Let g_0 be a Riemannian metric which is of the form

$$g_0 = dx^2 + g_{\partial M}(0)$$

near the boundary. Let $\tilde{e}(g_0, g)$ be the Chern-Simons class defined in [BZ], which satisfies the equation

$$d\tilde{e}(g_0, g) = e(g) - e(g_0), \tag{1.1}$$

where $e(g)$ denotes the Pfaffian form of g. Set, for $l \in [0, 1]$,

$$g_l = lg + (1 - l)g_0. \tag{1.2}$$

Let $\rho : \pi_1(M) \longrightarrow U(N)$ be a unitary representation. Denote by i the inclusion of ∂M into M.

Theorem 1.1 *Let $T = T(M, \rho)$ be the analytic torsion of $(M, \ \rho)$ with respect to the absolute boundary condition and $\tau = \tau(M, \rho)$ be the corresponding Reidemeister torsion. Then*

$$\ln(T/\tau) = \chi(\partial M, \rho) \ln 2 + \operatorname{rank} \rho \int_{\partial M} i^* \tilde{e}(g_0, g) + c \operatorname{rank} \rho \int_{\partial M} i^* \phi,$$

where

$$c = \int_0^\infty \int_0^1 \int_0^\infty x(e^{-(x-x')^2/4s} + e^{-(x+x')^2/4s})2e^{-(x+x')^2/4(1-s)} \, dx' \, ds \, dx$$

and ϕ is a differential form on M defined by the Berezin integral

$$\int^B \int_0^1 h_{ab} e^a \wedge \hat{e}^b \mathcal{R}'_0 e^{-\mathcal{R}} \, dl,$$

where $h_{ab}, \ 1 \leq a, b \leq n - 1$, is the second fundamental form of the boundary,

$$\mathcal{R}'_0 = \frac{1}{4} R_{0jkl} \, e^j \wedge \hat{e}^k \wedge \hat{e}^l$$

and

$$\mathcal{R} = \frac{1}{8} R_{ijkl} \, e^i \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l.$$

Remark. One also obtain a similar formula for the relative boundary condition, with only sign changes.

Remark. The reader should consult [LS] for examples where the these geometric contributions are shown to be nonzero, using the formulas given in [BG]. The article [BG] also contains a thorough discussion on the aboslute and relative boundary conditions.

An immediate application of this result is a gluing formula for the analytic torsion, since the gluing formula for R-torsion is well known.

Our strategy of proof is a very natural one. Namely we deform to the product case, where the difference of the analytic torsion and R-torsion is known by [L], [V]. The variation of the log of the ratio of the analytic torsion over the R-torsion is given by the constant term in certain asymptotic expansion of the heat kernel. To compute this term we employ the local index theorem technique, much in the spirit of Bismut-Zhang [BZ]. The new difficulty, however, is that the boundary condition introduces certain non-uniform behavior in the local heat asymptotic expansion, in the form of some Gaussian type concentration near the boundary. This difficulty is overcome by bringing in the technique of Melrose (Cf. [DM]). The non-uniform behavior of the heat kernel for the absolute boundary condition (say) is resolved by lifting it to a larger space obtained by performing certain blowup operation on the usual carrier of the heat kernel. This technique also effectively separates the interior and boundary contributions. The interior contribution is zero, just as in the closed case, while the

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boundary contribution comes from a model problem on the half tangent space of the manifold at the boundary.

2   Analytic torsion for manifold with boundary

In this section we recall the definition of the absolute and relative boundary conditions, their corresponding torsion invariants, while fixing our notations. We also state the variational formula for the log of the ratio of the analytic torsion over the R-torsion (from [RS1] and [C]), which is our starting point.

2.1   Absolute and relative boundary conditions

Let M be a compact manifold with boundary and g be a Riemannian metric on M. If we denote x the geodesic distance to the boundary ∂M, then, near the boundary,

$$g = dx^2 + g_{\partial M}(x),$$

where $g_{\partial M}(x)$ is a family of Riemannian metrics on the boundary. Let $\xi \to M$ be the flat bundle associated to a representation $\rho : \pi_1(M) \to U(k)$. Consider the Hodge Laplacian

$$\Delta = (d + \delta)^2 : \ C^\infty(M; \Lambda(M) \otimes \xi) \to C^\infty(M; \Lambda(M) \otimes \xi). \tag{2.3}$$

At the boundary we have the splitting

$$\Lambda(M) \otimes \xi|_{\partial M} = \Lambda(\partial M) \otimes \xi \oplus \Lambda(\partial M) \otimes \xi \tag{2.4}$$

corresponding to the decomposition for a form $\omega \in C^\infty(M; \Lambda(M) \otimes \xi)$:

$$\omega = \omega_t + dx \wedge \omega_n, \ \omega_t, \ \omega_n \in C^\infty(M; \Lambda(\partial M) \otimes \xi)$$

near the boundary. Define a linear map σ:

$$\sigma(\omega) = \omega_t - dx \wedge \omega_n.$$

Then σ is self adjoint and $\sigma^2 = 1$. Moreover the splitting (2.4) corresponds to the decomposition into the ± 1-eigenspace of σ.

From the splitting we define two projections

$$P_a : \ C^\infty(\partial M; \Lambda(M) \otimes \xi|_{\partial M}) \ \rightarrow \ C^\infty(\partial M; \Lambda(\partial M) \otimes \xi),$$
$$P_a(\omega) \ = \ \omega_n|_{\partial M};$$
$$P_r : \ C^\infty(\partial M; \Lambda(M) \otimes \xi|_{\partial M}) \ \rightarrow \ C^\infty(\partial M; \Lambda(\partial M) \otimes \xi),$$
$$P_r(\omega) \ = \ \omega_t|_{\partial M}.$$

i.e., P_a is the orthogonal projection onto the -1-eigenspace of σ and P_r the orthogonal projection onto the $+1$-eigenspace. The absolute and relative boundary conditions for the Hodge Laplacian are

$$\text{absolute}: \quad \begin{cases} P_a(\omega|_{\partial M}) = 0, \\ P_a(d\omega|_{\partial M}) = 0. \end{cases}$$

$$\text{relative}: \quad \begin{cases} P_r(\omega|_{\partial M}) = 0, \\ P_r(\delta\omega|_{\partial M}) = 0. \end{cases}$$

It is well known that these define elliptic boundary conditions for the Hodge Laplacian, with the corresponding cohomology being the absolute (resp. relative) cohomology of M. Hence the names.

To get a flavor of these boundary conditions, the special case when $M = \partial M \times \mathbb{R}_+$, and $g = dx^2 + g_{\partial M}$, where x is the variable of $\mathbb{R}_+$ and $g_{\partial M}$ is a metric on the boundary, deserves some elaboration. Using separation of variables, we can write

$$\omega = f(x)\omega_1^{\partial M} + g(x)dx \wedge \omega_2^{\partial M}, \tag{2.5}$$

where $\omega_1^{\partial M}$ and $\omega_2^{\partial M}$ are differential forms on ∂M. Then the absolute and relative boundary conditions are reduced to a combination of Dirichlet and Neumann boundary conditions:

$$\text{absolute}: \quad \begin{cases} g(0) = 0, \\ \frac{\partial f}{\partial x}(0) = 0. \end{cases} \tag{2.6}$$

$$\text{relative}: \quad \begin{cases} f(0) = 0, \\ \frac{\partial g}{\partial x}(0) = 0. \end{cases} \tag{2.7}$$

This greatly simplifies our computation of the model problem.

Back to the general discussion, it follows easily from the Stokes' Theorem that

$$(\Delta\omega, \theta) = (\omega, \Delta\theta) + \int_{\partial M} \omega \wedge *d\theta - \int_{\partial M} \theta \wedge *d\omega + \int_{\partial M} d^*\omega \wedge *\theta - \int_{\partial M} d^*\theta \wedge *\omega. \tag{2.8}$$

The boundary terms all vanish if both ω, θ satisfy the absolute (resp. relative) boundary condition. i.e. the absolute and relative boundary conditions are also self adjoint boundary conditions.

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Therefore, if $F_1(t, x, z)$, $F_2(t, z, y)$ are double forms on M that both satisfy the absolute (resp. relative) boundary condition, then (Cf. [C, p275])

$$(F_1(0, x, \cdot), F_2(t, \cdot, y)) - (F_1(t, x, \cdot), F_2(0, \cdot, y))$$
$$= \int_0^t (F_1(t - s, x, \cdot), (\partial_t + \Delta)F_2(s, \cdot, y))ds - \int_0^t ((\partial_t + \Delta)F_1(t - s, x, \cdot), F_2(s, \cdot, y))ds.$$

Thus if we denote

$$F_1 * F_2 = \int_0^t \int_M \langle F_1(t - s, x, z), F_2(s, z, y) \rangle dz \, ds, \tag{2.9}$$

then we have arrived at the so called Duhamel principle:

Proposition 2.1 *Let $e^{-t\Delta}$ denote the heat kernel of the Laplacian with the absolute (resp. relative) boundary condition. Let $F_1(t, x, y)$ be a double form on M which satisfies the absolute (resp. relative) boundary condition, and is the delta function at $t = 0$. Then*

$$e^{-t\Delta} = \sum_{m=1}^{\infty} (-1)^{m-1} F_m, \tag{2.10}$$

where $F_m = F_{m-1} * (\partial_t + \Delta)F_1$.

Proof. This follows by iterating the formula above. ∎

2.2 Analytic torsion and R-torsion

With elliptic self adjoint boundary conditions at our disposal the analytic torsions for manifolds with boundary can be defined using exactly the same formula as in the closed case. We only need to replace the usual heat kernel with the heat kernel corresponding to the appropriate boundary conditions. For simplicity it will be implicitly assumed that our heat kernels correspond to the absolute boundary condition, unless stated otherwise.

Thus, the analytic torsion $T(M, \rho)$ is defined in terms of the torsion zeta function

$$\log T(M, \rho) = \zeta_T'(0) \tag{2.11}$$

where $\zeta_T(s)$ is a meromorphic function of $s \in \mathbb{C}$, defined as follows. For $\Re s >> 0$

$$\zeta_T(s) = \frac{1}{\Gamma(s)} \sum_{p=1}^{\dim M} p(-1)^{p+1} \int_0^\infty t^s \text{Tr}\left(e^{-t\Delta_p} P^\perp\right) \frac{dt}{t}, \tag{2.12}$$

where Δ_p is the Laplacian on p-forms satisfying the absolute boundary conditions, and $P^\perp$ denote the projection onto the orthogonal complement of the harmonic forms. Using the standard elliptic theory, one sees that $\zeta_T(s)$ extends to a meromorphic

function of $s \in \mathbb{C}$ which is regular at $s = 0$. In particular, the right hand side of (2.11) is well-defined.

The natural $\mathbb{Z}$ grading on the space of differential forms induces a natural $\mathbb{Z}_2$ grading. We denote the corresponding supertrace Tr_s. With this notation formula (2.12) can be rewritten as

$$\zeta_T(s) = -\frac{1}{\Gamma(s)} \int_0^\infty t^s \mathrm{Tr}_s \left(N e^{-t\Delta} P^\perp \right) \frac{dt}{t},$$

where N denotes the so called number operator which mutiplies a differential p-form by p.

Similarly, the corresponding R-torsion $\tau(M, \rho)$ can be defined the same way as in the closed case using the absolute cochain complex.

The analytic torsion as defined above depends on the Riemannian metric g. So does the R-torsion. The variation of the torsion invariants with respect to the metric change is well studied. We quote the following result from [RS1], [C].

Proposition 2.2 *Let g_l be a family of Riemannian metrics ($l \in [0, 1]$), and $*_l$ the corresponding Hodge $*$-operator. Denote $T_l = T_l(M, \rho)$ (resp. $\tau_l = \tau_l(M, \rho)$) the analytic torsion (resp. R-torsion) with respect to g_l. If the unit normal of ∂M with respect to g_l is independent of the parameter l, then*

$$\frac{d}{dl} \log T_l/\tau_l = \lim_{t \to 0} \mathrm{Tr}_s(*_l^{-1} \frac{d*_l}{dl} e^{-t\Delta_l}) \tag{2.13}$$

Remark. In general the right hand side of (2.13) should be interpreted as the constant term in the asymptotic expansion of $\mathrm{Tr}_s(*_l^{-1}\frac{d*_l}{dl}e^{-t\Delta_l})$. In our case, however, all the negative powers of the asymptotic expansion drop out, as we will see from our proof, and the limit therefore is the genuine limit.

Thus, to prove our result, we deform to the product case. Namely, if we denote x the geodesic distance to the boundary ∂M with respect to the Riemannain metric g on M, then g can be written as

$$g = dx^2 + g_{\partial M}(x) \tag{2.14}$$

near the boundary. Now, let g_0 be a Riemannian metric which is of the form

$$g_0 = dx^2 + g_{\partial M}(0) \tag{2.15}$$

near the boundary. Set
$$g_l = lg + (1 - l)g_0. \tag{2.16}$$

We compute the right hand side of (2.13), using the local index theorem technique in the spirit of [BZ]. However, before that is possible, we first need to develop a thorough understanding of the uniform structure of the heat kernel for absolute (or relative) boundary condition.

3 The Uniform Structure of Heat Kernel

In this section we analyze the uniform structure of the heat kernel for elliptic boundary value problems of the type discussed above. The local index theorem technique dictates that we study the pointwise trace of the heat kernel. However, the elliptic boundary conditions introduce certain nonuniform behavior in the pointwise trace of the heat kernel in the form of some Gaussian type concentration at the boundary. We resolve this nonuniform behavior by studying the heat kernel in a bigger space, by introducing certain singular coordinates. In fact, we will construct a pseudodifferential calculus adapted to the situation and show that the heat kernel is certain element in this calculus. All the regularity and uniform property is contained in this statement.

3.1 Construction of the pseudodifferential calculus

Let $Z = M \times M \times \mathbb{R}_+$ be the usual carrier for the heat kernel. If we denote by (x, y) the local coordinates of M near ∂M, then (x, y, x', y', t) will be a local coordinates on Z. Our analysis of the heat kernel will be carried out in a blowup version of the space Z. For a space X, we denote by $\Delta(X)$ the diagonal in $X \times X$. Let $B_1 = \Delta(\partial M) \times \{0\}$ and $B_2 = \Delta(M) \times \{0\}$ be subspaces of Z. Also denote by S the parabolic bundle generated by dt. Consider the space [DM]

$$M_{h0}^2 = [Z; B_1, B_2, S] \tag{3.17}$$

obtained from Z by blowing up, parabolically in t, first B_1, then B_2. The blow down map is denoted by β_{h0}. That is, $\beta_{h0} : M_{h0}^2 \to Z$.

The new space M_{h0}^2 is a manifold with corner. For the moment let's look at the structures of the boundary hypersurfaces. There are three of them lying above $\{t = 0\}$. Among these we have B_{ff} from the blow up of B_1, called the front face; B_{tf} from the blow up of B_2, called the temporal face. These two are fibered over the submanifolds that are blown up. In fact B_{tf} can be viewed as a natural compactification of the rescaled tangent bundle ^0TM of M, whose sections are precisely vector fields vanishing at the boundary, as can be verified using suitable coordinates such as (3.20). Similarly B_{ff} is related to the half tangent bundle of M at ∂M as follows: the front face B'_{ff} resulting from the first blowup fibers over $B_1 \cong \partial M$, and in fact is canonically isomorphic to

$$\mathbb{R}_+ \times T^+_{\partial M} M \to \partial M, \tag{3.18}$$

with $T^+_{\partial M} M$ denoting the half tangent bundle of M at ∂M. Then B_{ff} is a suitable blowup of B'_{ff}, namely, if T is the coordinate of $\mathbb{R}_+$, $(s, \bar{Y}, y')$ the fiber coordinates of $T^+_{\partial M} M$, then B_{ff} is obtained by fiberwise blowing up $T = 0, s = 1, \bar{Y} = 0$.

The rest of the boundary hypersurfaces arises from the lift of those of Z under the blowdown map. More precisely we have the lift of the hypersurface $t = 0$ of Z,

denoted by $B_{tb}(M_{h0}^2)$; the lift of the hypersurface $x = 0$ of Z, $B_{lb}(M_{h0}^2)$; and the lift of $x' = 0$, $B_{rb}(M_{h0}^2)$. Note that B_{tf} only meets B_{ff}, B_{tb}.

In effect, the blowup is equivalent to introducing certain singular coordinates near the boundary. In the first blowup, the projective coordinates

$$T = t/x'^2, \ s = x/x', \ x', \bar{Y} = \frac{y - y'}{x'}, \ y' \tag{3.19}$$

will be valid coordinates near the front face, except at the corner with B_{rb}. After the second blowup, we can use the projective coordinates

$$T, \ X = \frac{s - 1}{\sqrt{T}} = \frac{x - x'}{\sqrt{t}}, \ Y = \frac{\bar{Y}}{\sqrt{T}} = \frac{y - y'}{\sqrt{t}}, \ x', \ y' \tag{3.20}$$

near the temporal face, except at the corner with B_{tb}.

We will denote by ρ the defining function for each boundary hypersurface, with the appropriate subscript attached. Thus ρ_{ff} will denote the defining function of B_{ff}, etc.

The pseudodifferential calculus will consists of operators whose kernels are defined on the space M_{h0}^2 and normalized with respect to the half-density

$$KD_{h0} = \rho_{ff}^{-(n+2)/2} \rho_{tf}^{-(n+3)/2} \Omega^{1/2}(M_{h0}^2), \tag{3.21}$$

where $\Omega^{1/2}$ denotes the standard half-density bundle. Let I denote the index set $\{k_1, k_2, k_3, k_4\}$. The space of operators with index set I is defined to be

$$\Psi_{h0}^I(M, \Lambda M \otimes \xi) = \rho_{ff}^{k_1} \rho_{tf}^{k_2} \rho_{lb}^{k_3} \rho_{rb}^{k_4} \rho_{tb}^{\infty} C^{\infty}(M_{h0}^2; \mathrm{Hom}(\Lambda M \otimes \xi) \otimes KD_{h0}).$$

To see how the operators in Ψ_{h0}^I act, consider the bilinear map from

$$\dot{C}_0^{\infty}([0, \infty) \times M; \Lambda M \otimes \xi \otimes \Omega^{1/2}) \times \dot{C}^{\infty}([0, \infty) \times M; \Lambda M \otimes \xi \otimes \Omega^{1/2})$$

to

$$\dot{C}^{\infty}(Z; \Lambda M \otimes \xi \otimes \Omega^{1/2})$$

defined by

$$\phi \hat{*}_t \psi = \int_0^{\infty} \phi(t + t', x) \psi(t', x') dt' \tag{3.22}$$

Lifting to M_{h0}^2 we can define

$$\langle A\psi, \phi \rangle = \int_{M_{h0}^2} A \cdot \beta_{h0}^*(\phi \hat{*}_t \psi). \tag{3.23}$$

It can be shown that these operators form a filtered algebra containing the heat kernel of an elliptic boundary value problem. However we do not need the full strength of this result and therefore will only be establishing those that we need in the following subsections.

3.2   Normal homomorphisms

Just like the classical pseudodifferential operators, the regularity property of elements of Ψ^I_{h0} is described by their symbols, except that there are more than just one (principal) symbol here, called the normal homomorphisms.

The normal homomorphism at B_{tf}, or the heat homomorphism, is defined by dividing out $T^{\frac{k_2}{2}}$ and restricting to B_{tf}:

$$N_{h,k_2}: \ \Psi^I_{h0} \to \rho^{k_1}_{ff}\rho^{k_3}_{lb}\rho^{k_4}_{rb}\rho^{\infty}_{tb}C^{\infty}(B_{tf}; KD_{h0}|_{B_{tf}}). \qquad (3.24)$$

Since elements of Ψ^I_{h0} vanish rapidly at B_{tb} the heat homomorphism simply describes the leading term at B_{tf}. We note that B_{tf} meets only B_{ff} and B_{tb}. Moreover, since B_{tf} is the natural compactification of ^0TM, $KD_{h0}|_{B_{tf}}$ is canonically isomorphic to the fiber density bundle of ^0TM and thus the heat homomorphism can be rewritten as

$$N_{h,k_2}: \ \Psi^I_{h0} \to \rho^{k_1}_{ff}\mathcal{S}(^0TM; \Omega_{\text{fiber}}).$$

Here $\mathcal{S}(^0TM; \Omega_{\text{fiber}})$ denotes the space of smooth fiber densities which are of Schwarz class along fibers of ^0TM.

The normal homomorphism at B_{ff} describes the leading term at B_{ff}. Thus we divide out $\rho^{k_1}_{ff}$ and restrict to B_{ff}:

$$N_{f,k_1}: \ \Psi^I_{h0} \to \rho^{k_2}_{tf}\rho^{k_3}_{lb}\rho^{k_4}_{rb}\rho^{\infty}_{tb}C^{\infty}(B_{ff}; KD_{h0}|_{B_{ff}}). \qquad (3.25)$$

By the previous discussion on the structure of B_{ff} (see (3.18)) the range of N_{f,k_1} is isomorphic to the space of smooth functions on $\mathbb{R}_+ \times T^+_{\partial M}M$, except for conormal concentration along the submanifold $s = 1, \bar{Y} = 0$ at $T = 0$, with the order of vanishing at the boundaries $s = 0$ and $s = \infty$ prescribed by the index set, and vanishing rapidly at $\bar{Y}$-infinity. In other words, the range of N_{f,k_1} is the (restricted) fiberwise heat calculus on $T^+_{\partial M}M$.

For an operator in Ψ^I_{h0}, its values under the normal homomorphisms will be called its normal operators. Individually, each normal homomorphism is surjective. However the normal operators for an element of Ψ^I_{h0} have to agree at the common corners. These are the compatibility conditions. On the other hand, since essentially just smooth functions are involved, the compatibility conditions are clearly the only obstructions to the existence of operators with the prescribed normal operators. This fact will be used in constructing heat kernels as an element of Ψ^I_{h0}.

3.3   Compositions

Another ingredient in constructing heat kernels as an element of Ψ_{h0} will be the composition properties of elements of Ψ_{h0}. We first study the composition of these operators with differential operators.

Proposition 3.1 *Let $A \in \Psi_{h0}^I(M, \Lambda M \otimes \xi)$. If V is any smooth vector field on M, and $T = t/x'^2$, then we have*

$$(t^{\frac{1}{2}}V) \circ A \in \Psi_{h0}^I(M, \Lambda M \otimes \xi)$$

and

$$N_{f,k_1}(t^{\frac{1}{2}}V \circ A) = T^{\frac{1}{2}}\sigma_1(V)N_{f,k_1}(A), \quad N_{h,k_2}(t^{\frac{1}{2}}V \circ A) = \sigma_1(V)N_{h,k_1}(A). \quad (3.26)$$

Here $\sigma_1(V)$ denotes the usual symbol of a vector field. Furthermore,

$$(t\partial_t \circ A) \in \Psi_{h0}^I(M, \Lambda M \otimes \xi)$$

and

$$N_{f,k_1}(t\partial_t \circ A) = T\partial_T N_{f,k_1}(A), N_{h,k_2}(t\partial_t \circ A) = -\frac{1}{2}(R+n+2+k_2)N_{h,k_2}(A). \quad (3.27)$$

Proof. We first assume that ρ is the trivial representation, i.e. ξ is the trivial bundle. Clearly we can restrict our attention to a region near the front and temporal faces. Therefore we assume that A is supported near such a region. Further, by partition of unity, we can assume that A is either supported near the front face but away from the temporal face, or is supported near the temporal face.

For the first case, we get to use the projective coordinates (3.19), But it is more convenient to use the project coordinate

$$t, \ X = \frac{x}{t^{\frac{1}{2}}}, \ X' = \frac{x'}{t^{\frac{1}{2}}}, \ Y = \frac{y-y'}{t^{\frac{1}{2}}}, \ y', \quad (3.28)$$

which is valid near the front face, except at the corner, where $X, X', Y = \infty$. In this coordinate $A \in \Psi_{h0}^I(M)$ can be written as $A = t^{\frac{k_1}{2} - \frac{n+2}{4}}a(t, X, X', Y, y')|dtdXdX'dYdy'|^{\frac{1}{2}}$, where a is a smooth function of $(t^{\frac{1}{2}}, X, X', Y, y')$ and vanishes rapidly as $X, X', Y \to \infty$. In this representation $N_{f,k_1}(A) = a(0, X, X', Y, y')|dXdX'dYdy'|^{\frac{1}{2}}$.

Now if $\phi = \phi_0|dtdxdy|^{\frac{1}{2}}$ and $\psi = \psi_0|dtdxdy|^{\frac{1}{2}}$ we have

$$\langle A\psi, \phi \rangle = \int_{M_h^2} t^{\frac{k_1}{2}}a\beta_h^*(\phi_0 \hat{*}_t \psi_0)|dtdXdX'dYdy'|.$$

Also, if we let $V = v_1(x,y)\partial_x + v_2(x,y)\partial_y$ be a smooth vector field and V' denote its transpose:

$$\langle V\psi, \phi \rangle = -\langle \psi, V'\phi \rangle,$$

then $V' = \partial_x \circ v_1 + \partial_y \circ v_2$, and $\beta_h^*(t^{\frac{1}{2}}V') = t^{\frac{1}{2}}\partial_x(v_1) + t^{\frac{1}{2}}\partial_y(v_2) + v_1\partial_X + v_2\partial_Y$.

A computation similar to that of [DM, p44] then shows that

$$(t^{\frac{1}{2}}V) \circ A = t^{\frac{k_1}{2} - \frac{n+2}{4}}(v_1\partial_X(a) + v_2\partial_Y(a) + t^{\frac{1}{2}}(\partial_x v_1)a + t^{\frac{1}{2}}(\partial_y v_2)a)|dtdXdX'dYdy'|^{\frac{1}{2}}$$

and therefore $(t^{\frac{1}{2}}V) \circ A \in \Psi^I_{h0}(M)$, with

$$N_{f,k_1}(t^{\frac{1}{2}}V \circ A) = (v_1\partial_X + v_2\partial_Y)a(0, X, X', Y, y')|dX dX' dY dy'|^{\frac{1}{2}}. \qquad (3.29)$$

Now we change the coordinate from (3.28) to (3.19) and get

$$\partial_X = T^{\frac{1}{2}}\partial_s, \qquad\qquad \partial_Y = T^{\frac{1}{2}}\partial_{\bar{Y}}, \qquad (3.30)$$

which, when plugged into (3.29), immediately gives

$$N_{f,k_1}(t^{\frac{1}{2}}V \circ A) = T^{\frac{1}{2}}\sigma_1(V)N_{f,k_1}(A).$$

Similarly we have $t\partial_t \circ A \in \Psi^I_{h0}(M)$ and

$$N_{f,k_1}(t\partial_t \circ A) = [\frac{k_1}{2} - \frac{1}{2}(n+1+X\partial_X + X'\partial'_X + Y\partial_Y)]N_{h,-j}(A). \qquad (3.31)$$

But changing to the coordinate (3.19) we see

$$N_{f,k_1}(t\partial_t \circ A) = T\partial_T N_{f,k_1}(A).$$

For the second case when the support of A is near the temporal face, the normal operators can be obtained in the same way using the projective coordinates (3.20).

Finally, if ρ is not trivial, by linearity, we can assume that ϕ, ψ and A are supported in a small neighborhood where ξ is trivialized by an orthonormal basis $\{s_i\}$. Write

$$\phi = \phi_i s_i, \ \psi = \psi_i s_i, \ A = A_{ij}s^*_i \otimes s_j.$$

Then

$$\begin{aligned} \langle (V \circ A)\psi, \phi \rangle &=& -\langle A\psi, \nabla_V\phi \rangle \\ &=& -\langle A_{ij}\psi_i, V\phi_j + \Gamma_{kj}(V)\phi_k \rangle, \end{aligned}$$

where $\Gamma_{kj}(V) = \langle \nabla_V s_i, s_j \rangle$. This reduces to the scalar case and the connection produces only a lower order term. ∎

We now consider the composition of elements of $\Psi^I_{h0}(M; \Lambda M \otimes \xi)$. For simplicity we will not consider the full composition properties, only the composition of the following Voltera type operators, which suffices for our purpose. Thus if $\mathcal{E} = (E_{ff}, E_{lb}, E_{rb})$ is an index family for M^2_{h0}, assumed trivial at B_{tf} and B_{tb} then let

$$\Psi^{-\infty,\mathcal{E}}_{h0}(M, \Omega^{\frac{1}{2}}) = \Psi^{E_{ff},\infty,E_{lb},E_{rb}}_{h0}(M, \Lambda M \otimes \xi)$$

be the space of polyhomogeneous conormal distributions on M^2_{h0} which vanish rapidly at B_{tf} and B_{tb} and have expansions at B_{ff}, B_{lb} and B_{rb} with exponents from E_{ff}, E_{lb} and E_{rb} respectively.

Proposition 3.2 *We have*

$$\Psi_{h0}^{-\infty,\mathcal{E}}(M,\Omega^{\frac{1}{2}}) \circ \Psi_{h0}^{-\infty,\mathcal{F}}(M,\Omega^{\frac{1}{2}}) \subset \Psi_{h0}^{-\infty,\mathcal{G}}(M,\Omega^{\frac{1}{2}})$$

where

$$
\begin{aligned}
G_{ff} &= [E_{ff}+F_{ff}]\cup[E_{lb}+F_{rb}+n+2] \\
G_{lb} &= [E_{ff}+F_{lb}]\cup[E_{lb}] \\
G_{rb} &= [F_{ff}+E_{rb}]\cup[F_{rb}]\,.
\end{aligned}
$$

Proof. Since the operators here all have their Schwartz kernel vanishing rapidly at B_{tf}, we might as well consider them as living in

$$M_0^2 = [Z; B_1, S].$$

The proof relies on lifting everything to a triple space. For this reason we consider

$$W = \mathbb{R}_+^2 \times M^3 = \{(t,t',m,m',m'') \mid t \geq t'\},$$

and the following submanifolds of W:

$$B_F = \{(t,0,m,m',m')\}, \quad B_S == \{(t,t,m,m,m'')\}, \quad B_C = \{(0,t',m,m',m)\},$$

together with

$$B_T = \{(0,0,m,m,m)\},$$

which is the intersection of the all three. Let S_F, S_S, S_C be the lift of the parabolic bundle $S = \{dt\}$ to the corresponding submanifolds B_F, B_S, B_C and $S_T = \{dt, dt'\}$ the parabolic bundle on B_T. Then the triple space W_0 is obtained from W by appropriate blowups:

$$W_0 = [W; B_T, S_T; B_F, S_F; B_S, S_S; B_C, S_C].$$

Once again we obtain a manifold with corner. We label the hypersurfaces created by the blowups by fF, ff, sf, cf respectively, and those from the lift by fb, sb, cb. The triple space W_0 is related to the double space M_0^2 by the b-fibrations

$$\pi_F,\ \pi_S,\ \pi_C:\ W_0 \to M_0^2,$$

corresponding to the projections $p_F,\ p_S,\ p_C:\ W \to Z$,

$$
\begin{aligned}
p_F(t,t',m,m',m'') &= (t',m',m''), \\
p_S(t,t',m,m',m'') &= (t-t',m,m'), \\
p_C(t,t',m,m',m'') &= (t,t',m,m'').
\end{aligned}
$$

Let $f:\ X \longrightarrow Y$ be a b-map between manifolds with corners, i.e., a C^∞ map such that if $\rho_i' \in C^\infty(Y)$, $i = 1,\ldots,N'$ are defining functions for the boundary

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hypersurfaces of Y and $\rho_j \in C^\infty(X)$, $j = 1, \ldots, N$ are defining functions for the boundary hypersurfaces of X then

$$f^*\rho_i' = a_i \prod_{j=1}^{k(i,j)} \rho_j^{k(i,j)}, \ 0 < a_i \in C^\infty(X).$$

The non-negative integers $k(i,j)$ are the boundary exponents of f. In the following table we computed all boundary exponents of the b-fibrations π_F, π_S, π_C.

		fF	ff	sf	cf	fb	sb	cb
	ff	1	1	0	0	0	0	0
π_F	lb	0	0	1	0	0	0	1
	rb	0	0	0	1	0	1	0
	ff	1	0	1	0	0	0	0
π_S	lb	0	0	0	1	1	0	0
	rb	0	1	0	0	0	0	1
	ff	1	0	0	1	0	0	0
π_C	lb	0	0	1	0	1	0	0
	rb	0	1	0	0	0	1	0
	ν	0	0	0	n+2	0	0	0

Also in the Table there is a 'density row', labelled 'ν' which is important in the description of the composition results. These exponents are fixed by the natural identification of density bundles:

$$(\pi_F)^*KD \otimes (\pi_S)^*KD \otimes (\pi_C)^*(KD') \otimes |dt|^{\frac{1}{2}} \cong \prod_{F=fF,ff,sf,cf,fb,sb,cb} \rho_F^{\nu_F} \cdot \Omega(W_0).$$

Here KD' is the half-density bundle with the opposite weighting to KD so that $KD' \otimes KD \cong \Omega$.

Let $A \in \Psi_{h0}^{-\infty,\mathcal{E}}(M; \Omega^{\frac{1}{2}})$, $B \in \Psi_{h0}^{-\infty,\mathcal{F}}(M, \Omega^{\frac{1}{2}})$. The composition $C = A \circ B$ can be written in terms of their Schwartz kernels via the b-fibrations π_O, $O = F, S, C$:

$$\kappa_C KD' = (\pi_C)_*[(\pi_S)^*\kappa_A \cdot (\pi_F)^*\kappa_B \cdot (\pi_C)^*(KD') \cdot (\tilde{\pi}_C)^*(|dt'|^{\frac{1}{2}})],$$

where $\tilde{\pi}_C = \pi_C^2 \circ \beta_0^3 : W_0 \to Z$ and β_0^3 is the blow down map.

By the push-forward theorem

$$C \in \Psi_{h0}^{-\infty,\mathcal{G}}(M, \Omega^{\frac{1}{2}})$$

for some index set $\mathcal{G}$. The index set $\mathcal{G} = (G_{ff}, G_{lb}, G_{rb})$ can be computed by the Mellin transform

$$\langle \kappa_C \cdot KD', \rho_{ff}^{z_1} \rho_{lb}^{z_2} \rho_{rb}^{z_3} \rangle = \langle (\pi_F)^*(\tilde{\kappa}_A \cdot (\pi_S)^*(\tilde{\kappa}_B) \cdot \prod_F \rho_F^{\nu_F} \cdot \Omega(W_0), (\pi_C)^*(\rho_{ff}^{z_1} \rho_{lb}^{z_2} \rho_{rb}^{z_3}) \rangle,$$

where $\kappa_A = \tilde{\kappa}_A \cdot KD$, $\kappa_B = \tilde{\kappa}_B \cdot KD$. From the table we obtain the formulas for our index set. ∎

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3.4 Uniform structure of heat kernel

The proceeding construction enables us to prove the following theorem, which gives the uniform structure of the heat kernel for absolute (or relative) boundary condition.

Theorem 3.3 *If we denote by $e^{-t\Delta}$ the heat kernel of the Hodge Laplacian for the absolute (or relative) boundary condition, then*

$$e^{-t\Delta} \in \Psi_{h0}^{2,2,0,0}(M, \Lambda M \otimes \xi).$$

Moreover the normal operators of $e^{-t\Delta}$ at the front face and the temporal face are explicitly computable.

Proof. Our proof is actually constructive. Namely we construct a (unique) $H \in \Psi_{h0}^{I}(M; \Lambda M \otimes \xi)$ where $I = \{2, 2, 0, 0\}$ such that

$$t(\partial_t + \Delta)H = 0 \mod \Psi_{h0}^{I'}(M; \Lambda M \otimes \xi) \tag{3.32}$$

for $I' = \{3, 3, 1, 1\}$, and

$$N_{h,2}(H) = \mathrm{Id}, \tag{3.33}$$

and H satisfies the boundary condition

$$P_a(H|_{B_{lb}}) = 0, \ \ P_a(dH|_{B_{lb}}) = 0.$$

Equations (3.3) and (3.33) translate to conditions on the normal operators of H by using Proposition 3.1:

$$({}^0\sigma_2(t\Delta) - \frac{1}{2}(R+n))N_{h,2}(H) = 0, \ \int_{\text{fiber}} N_{h,2}(H) = \mathrm{Id}, \tag{3.34}$$

$$(\partial_T + \Delta_E)N_{f,2}(H) = 0. \tag{3.35}$$

Finally the boundary condition translates into the same type boundary condition for (3.35).

The first equation is a fiber by fiber differential equation and can be solved uniquely subject to the integral condition. Furthermore because of the compatibility condition this fixes the integral conditions for (3.35). Thus the normal operator $N_{h,2}(H)$ is necessarily the heat kernel for the elliptic boundary condition on the half tangent space. These two operators have the same indicial family, so using the existence part of the compatibility it follows that there is an element $G_1 \in \Psi_{h0}^{I}(M; \Lambda M \otimes \xi)$ satisfying the symbolic conditions (3.34), (3.35).

This first approximation therefore satisfies

$$t(\partial_t + \Delta)G_1 = -R_1, \ \ R_1 \in \Psi_{h0}^{3,3,1,1}(M; \Lambda M \otimes \xi). \tag{3.36}$$

Thus G_1 is already a parametrix. We now modify G_1. Using the heat calculus we can find a $G_0 \in \Psi_{h0}^{2,2,\infty,\infty}(M, \Lambda M \otimes \xi)$ such that

$$t(\partial_t + \Delta)G_0 = t\mathrm{Id} - tR_0,$$

where $R_0 \in \Psi_{h0}^{0,\infty,\infty,\infty}(M, \Lambda M \otimes \xi)$.

It follows that there is a correction term $G_0' \in \Psi_{h0}$ such that the modification of the parametrix $G_2 = G_1 - G_0'$ still has the correct normal operator and is a parametrix in the strong sense that

$$t(\partial_t + \Delta)G_2 = t\mathrm{Id} - tR_2,$$

where $R_2 \in \Psi_{h0}^{\infty,3,1,1}$.

By Proposition 3.2, $(R_2)^k \in \Psi_{h0}^{\infty,3k,1,1}$. Thus the Neumann series $\sum_{k=0}^{\infty}(R_2)^k$ can be summed modulo a term vanishing rapidly at B_{ff}, i.e. there exists $S' \in \Psi_{h0}^{\infty,1,1}$ such that

$$(Id - S')(\mathrm{Id} - R_2) = \mathrm{Id} - R_3, \quad R_3 \in \Psi_{h0}^{\infty,\infty,1,1}.$$

Thus $\mathrm{Id} - R_3$ can be inverted with an operator of the same type. It follows that $\mathrm{Id} - R_2$ has a two-side inverse $\mathrm{Id} - S$, $S \in \Psi_{h0}^{-\infty,1,1}(M, F)$. This in turn means we have $\exp(-t\Delta) = G_1(Id - S) = G_1 - G_1 \circ S$ and we obtain our result. ∎

In the discussion below we will be concerned with the trace of the elements of $\Psi_{h0}^{k,k}(M;E) = \Psi_{h0}^{k,k,0,0}(M;E)$. The elements of $\Psi_{h0}^{k,k}(M;E)$ are families of smoothing operators on M, hence trace class. By Lidsky's theorem the trace is the integral over the diagonal of the pointwise trace of the kernel. Consider the diagonal $\Delta(M) \times \mathbb{R}_+$ of Z, which lifts to an embedded submanifold of M_{h0}^2. Let $i : \Delta_{h0} \to M_{h0}^2$ be the embedding of the lifted diagonal. We have

$$\Delta_{h0} \cong M_{h0} = [M \times \mathbb{R}_+; \partial M \times \{0\}, S] \tag{3.37}$$

which blows down to $M \times \mathbb{R}_+$. Restricted to the lifted diagonal, the kernels of the elements of $\Psi_{h0}^{k,k}(M;E)$ can be interpreted as a density:

$$Hom(E) \otimes \Omega^{1/2}(M_{h0}^2)|_{\Delta_{h0}} \cong hom(E) \otimes \Omega(M_{h0}) \otimes \Omega^{\frac{1}{2}}(\mathbb{R}_+).$$

Thus the trace of $A \in \Psi_{h0}^{2,2,0,0}(M;E)$ is the push-forward to $\mathbb{R}_+$ of the density

$$(\mathrm{tr}\, A)|_{\Delta_{h0}} \in C^{\infty}(M_{h0}; \Omega(M_{h0}) \otimes \Omega^{\frac{1}{2}}(\mathbb{R}_+)).$$

Lemma 3.4 *As a map*

$$\mathrm{Tr}:\ \Psi_{h0}^{k,k}(M;E) \to t^{-\frac{n-k+2}{2}}\big(C^{\infty}(\mathbb{R}_+, \Omega^{\frac{1}{2}}(\mathbb{R}_+)) + t^{\frac{1}{2}}C^{\infty}(\mathbb{R}_+, \Omega^{\frac{1}{2}}(\mathbb{R}_+)) + O(t\log t)\big).$$

i.e.

$$\mathrm{Tr}(A) = t^{-\frac{n-k+2}{2}}\big(r_A(t) + t^{\frac{1}{2}}s_A(t) + O(t\log t)\big)\,|dt|^{\frac{1}{2}},$$

where $r_A(t)$, $s_A(t)$ are smooth functions of t. Moreover the leading terms can be computed by

$$r_A(0) = \int_M \operatorname{tr}(N_{h,k}(A)), \quad s_A(0) = \int_{B_f} \operatorname{tr}(N_{f,k}(A)), \tag{3.38}$$

where $M \to {}^0TM$ is the 0-section and B_f is the front face of M_{h0}.

Proof. This follows from the pushforward theorem, using Mellin transform. Namely, if we denote by β' the composition of the blowdown map $\beta_1 : M_{h0} = [M \times \mathbb{R}_+; \partial M \times \{0\}, S] \to M \times \mathbb{R}_+$ and the projection onto the second factor $M \times \mathbb{R}_+ \to \mathbb{R}_+$, we have

$$\langle (\beta')_*((\operatorname{tr} A)|_{\Delta_{h0}}),\ t^z |dt|^{\frac{1}{2}} \rangle = \langle (\operatorname{tr} A)|_{\Delta_{h0}},\ (\beta')^*(t^z |dt|^{\frac{1}{2}}) \rangle.$$

Now

$$
\begin{aligned}
KD|_{\Delta_{h0}} \otimes (\beta')^*(|dt|^{\frac{1}{2}}) &= (\beta_1)^*(t^{-\frac{n+2}{2}} \Omega^{\frac{1}{2}}(M^2 \times \mathbb{R}_+) \otimes \Omega^{\frac{1}{2}}(\mathbb{R}_+)|_{\Delta(M \times \mathbb{R}_+)}) \\
&= \rho_{ff}^{-n} \rho_{tf}^{-n-1} \Omega(M_{h0}),
\end{aligned}
$$

which gives us the desired result. ∎

4 Clifford Structure and Cancellation

In order to prove our main result we need to modify our previous construction by incorporating the local index theorem type computation of Bismut-Zhang. A crucial ingredient is the Clifford structure of the exterior algebra, which we recall now.

4.1 Clifford Structure of the Exterior Algebra

If E is a finite dimenional vector space of dimension n, the exterior algebra $\Lambda(E^*)$ is naturally $\mathbb{Z}$-graded, which induces a natural $\mathbb{Z}_2$-grading. If $A \in \operatorname{End}(\Lambda(E^*))$, we let $\operatorname{Tr}_s(A)$ denote the supertrace of A.

Now assume that E is equipped with an inner product g. If $e \in E$, let $e^* \in E^*$ correspond to e by the metric g. Set

$$
\begin{aligned}
c(e) &= e^* \wedge -i_e, \\
\hat{c}(e) &= e^* \wedge +i_e.
\end{aligned}
$$

These define elements of $\operatorname{End}(\Lambda(E^*))$. Moreover, we have the following commutator relations

$$
\begin{aligned}
c(e)c(e') + c(e')c(e) &= -2\langle e, e' \rangle, \\
\hat{c}(e)\hat{c}(e') + \hat{c}(e')\hat{c}(e) &= 2\langle e, e' \rangle, \\
c(e)\hat{c}(e') + \hat{c}(e')c(e) &= 0.
\end{aligned}
\tag{4.39}
$$

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From (4.39), we see that c and $\hat{c}$ extend to representations of the Clifford algebra of E. Furthermore, one verifies that $\mathrm{End}(\Lambda(E^*))$ is generated as an algebra by 1 and the $c(e), \hat{c}(e)$'s.

Let $e_1, \cdots, e_n$ be an orthonormal basis of E, and $e^1, \cdots, e^n$ the dual basis of E^*. A simple but essential algebraic fact is the following result which we quote from [BZ, Proposition 4.9].

Lemma 4.1 *Among the monomials in the $c(e), \hat{c}(e)$'s, only $c(e_1)\hat{c}(e_1) \cdots c(e_n)\hat{c}(e_n)$ has a nozero supertrace, which is*

$$\mathrm{Tr}_s[c(e_1)\hat{c}(e_1) \cdots c(e_n)\hat{c}(e_n)] = (-2)^n.$$

4.2 Supertrace and Cancellation

The above discussion applied fiberwise to TM shows that $\mathrm{End}(\Lambda^* M)$ is generated as an algebra by 1 and the $c(e), \hat{c}(e)$'s for any local orthonormal basis e. This gives a natural filtration of $\mathrm{End}(\Lambda^* M)$, which we will exploit by making a global rescaling of $\mathrm{End}(\Lambda^* M \otimes \xi)$ near the front and temporal faces of M_{h0}^2. Since this is localized near the diagonal, the vector bundle ξ does not appear in the discussion and we happily suppress its presence.

Denote by

$$F^0 \subset F^1 \subset \cdots \subset F^n$$

the natural filtration on $\mathrm{End}(\Lambda^* M)$ induced by the Clifford structure introduced above, i.e., F^j is the subspace generated by monomials of length $\leq j$ in $c(e), \hat{c}(e)$. Its associated graded algebra $G = \oplus G^i$, $G^i = F^i/F^{i-1}$, is naturally isomorphic to $\Lambda(T^*M) \otimes \Lambda(T^*M)$ (Cf. (4.39)). We distinguish the elements of the second copy of $\Lambda(T^*M)$ from the first by putting a hat on them. For example $\mathcal{R} = \frac{1}{8}R_{ijkl}\, e^i \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l \in \Lambda(T^*M) \otimes \Lambda(T^*M)$.

Recall that the Berezin integral [BZ] takes elements of $\Lambda(T^*M) \otimes \Lambda(T^*M)$ and sends them to $\Lambda(T^*M)$:

$$\int^B : \ \Lambda(T^*M) \otimes \Lambda(T^*M) \longrightarrow \Lambda(T^*M) \tag{4.40}$$

The Berezin integral is intimately related to the supertrace and cancellation mechanism.

Applying the construction in [DM] we obtain a rescaled version of the pseudodifferential calculus constructed above, $\Psi_{h0,G}^I(M; \Lambda^* M \otimes \xi)$. For example, $\Psi_{h0,G}^{2,2,0,0}(M; \Lambda^* M \otimes \xi)$ consists of those elements of $\Psi_{h0}^{2,2,0,0}(M; \Lambda^* M \otimes \xi)$ which have the following asymptotic expansion near either the front or temporal face F:

$$k = \rho_{ff}^2 \rho_{tf}^2 \sum_{j=0}^n \rho_F^{\frac{j}{2}} k_j, \quad k_j \in C^\infty(M_{h0}^2, F^j \otimes KD). \tag{4.41}$$

The normal operators $N_{h,k,G}$, $N_{f,k,G}$ of elements of $\Psi^I_{h0,G}(M; \Lambda^* M \otimes \xi)$ will now be taking values in the associated graded algebra $G \cong \Lambda(T^*M) \otimes \Lambda(T^*M)$.

The advantage of the modification comes from the following cancellation result. (Note the disappearance of the singular factor $t^{-\frac{n}{2}}$.)

Lemma 4.2 *As a map*

$$\mathrm{Tr}_s : \ \Psi^{k,k,0,0}_{h0,G}(M; \Lambda^* M \otimes \xi) \to t^{\frac{k-2}{2}}(C^\infty(\mathbb{R}_+, \Omega^{\frac{1}{2}}(\mathbb{R}_+)) + t^{\frac{1}{2}}C^\infty(\mathbb{R}_+, \Omega^{\frac{1}{2}}(\mathbb{R}_+)) + O(t \log t)).$$

i.e.
$$\mathrm{Tr}_s(A) = t^{\frac{k-2}{2}}(r_A(t) + t^{\frac{1}{2}} s_A(t) + O(t \log t)) |dt|^{\frac{1}{2}},$$

where $r_A(t)$, $s_A(t)$ are smooth functions of t. Moreover the leading terms can be computed by

$$r_A(0) = \int_M \int^B N_{h,k,G}(A), \quad s_A(0) = \int_{B_f} \int^B N_{f,k,G}(A). \tag{4.42}$$

Proof. This comes from the definition and the main algebraic fact Lemma 4.1. ∎

With this in mind we have the following significant refinement on the structure of the heat kernel.

Theorem 4.3 *The heat kernel $e^{-t\Delta}$ of the Hodge Laplacian for the absolute (or relative) boundary condition, is an element of the rescaled heat calculus:*

$$e^{-t\Delta} \in \Psi^{2,2,0,0}_{h0,G}(M, \Lambda^* M \otimes \xi).$$

Moreover the normal operators of $e^{-t\Delta}$ at the front face and the temporal face are given by

$$\begin{aligned}
N_{h,2,G} &= (4\pi)^{-\frac{n}{2}} \exp(-|v|^2) \exp(-\mathcal{R}), \\
N_{f,2,G} &= \exp(-T(\Delta_E + \mathcal{R})),
\end{aligned}$$

where $\mathcal{R} = \frac{1}{8} R_{ijkl}\, e^i \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l$ and Δ_E is the fibrewise Laplacian on the inward pointing half of tangent bundle TM with the absolute (relative respectively) boundary condition.

Proof. We only need to adapt the previous argument to the rescaling. According to the Bochner-Weitzenböck formula,

$$\Delta = \Delta^c - R_{ijkl}\mathrm{ext}(e_i)\mathrm{int}(e_j)\mathrm{ext}(e_k)\mathrm{int}(e_l).$$

Here $\Delta^c = \nabla^*\nabla$ is the connection Laplacian; with respect to any local orthonormal frame of TM it is
$$\Delta^c = -\sum_i (\nabla^2_{e_i} - \nabla_{\nabla_{e_i} e_i}).$$

Expressing everything in terms of $c(e)$ and $\hat{c}(e)$, we have

$$\Delta = \Delta^c + c(R) - \frac{1}{4}S,$$

with S the scalar curvature and

$$c(R) = \frac{1}{8}R_{ijkl}\,c(e_i)c(e_j)\hat{c}(e_k)\hat{c}(e_l).$$

Now since each Clifford element carries a weight of $\rho^{\frac{1}{2}}$ at both the temporal and the front faces, it follows then for $A \in \Psi^I_{h0,G}(M;\Lambda^* M \otimes \xi)$, $tc(R) \circ A \in \Psi^I_{h0,G}(M;\Lambda^* M \otimes \xi)$, and

$$N_{h,G}(tc(R) \circ A) = \mathcal{R}N_{h,G}(A), \quad N_{f,G}(tc(R) \circ A) = T\mathcal{R}N_{f,G}(A).$$

Finally, for the connection, we note that

$$\nabla_{e_i} = \Gamma^k_{ij}\text{ext}(e_k)\text{int}(e_j) = \frac{1}{4}\Gamma^k_{ij}(c(e_k) + \hat{c}(e_k))(\hat{c}(e_j) - c(e_j))$$

again produces only lower order terms. ∎

4.3 The Model Problem

Our model problem is to solve in $\mathbb{R}^n_+$ for the heat kernel of $\Delta + \mathcal{R}$, where $\mathcal{R} = \frac{1}{8}R_{ijkl}\,e^i \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l$, subject to the, say, absolute boundary conditions. More specifically, let $(x, y) \in \mathbb{R}_+ \times \mathbb{R}^{n-1} = \mathbb{R}^n_+$ and $e_0 = \partial_x$. Then we need to solve

$$\begin{cases} (\partial_t + \Delta + \mathcal{R})K = 0 \text{ in } \mathbb{R}^n_+ \\ K|_{t=0} = I \\ e^0 \wedge i_{e_0}K|_{x=0} = 0 \\ i_{e_0}e^0 \wedge \frac{\partial K}{\partial x}|_{x=0} = 0. \end{cases} \tag{4.43}$$

Lemma 4.4 *Let* $\mathcal{R}_0 = \frac{1}{4}R_{0jkl}\,e^0 \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l$. *We have*

$$\left[\mathcal{R},\ i_{e_0}e^0 \wedge\right] = \mathcal{R}_0, \quad \left[\mathcal{R},\ e^0 \wedge i_{e_0}\right] = -\mathcal{R}_0. \tag{4.44}$$

Proof. This is a straightforward computation using

$$i_{e_0}e^0 \wedge + e^0 \wedge i_{e_0} = 1.$$

In fact,

$$\begin{aligned} \mathcal{R}(i_{e_0}e^0 \wedge) &= \left(\frac{1}{8}\sum_{i,j \neq 0}R_{ijkl}\,e^i \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l + \mathcal{R}_0\right)(i_{e_0}e^0 \wedge) \\ &= (i_{e_0}e^0 \wedge)\mathcal{R} + \mathcal{R}_0 \end{aligned}$$

The other formula is obtained similarly. ■

Now, let

$$K_D(t) = \frac{1}{(4\pi t)^{n/2}}(e^{-(x-x')^2/4t} - e^{-(x+x')^2/4t})e^{-|y-y'|^2/4t},$$

$$K_N(t) = \frac{1}{(4\pi t)^{n/2}}(e^{-(x-x')^2/4t} + e^{-(x+x')^2/4t})e^{-|y-y'|^2/4t}$$

be the Dirichlet and Neumann heat kernels respectively. Also, for $k = k(t, x, y, x', y')$, $\tilde{k} = \tilde{k}(t, x, y, x', y')$, their convolution is

$$k * \tilde{k} = \int_0^t \int_0^\infty \int_{\mathbb{R}^{n-1}} k(s, x, y, x', y')\tilde{k}(t - s, x', y', x'', y'') \, dt \, dx' \, dy'. \qquad (4.45)$$

Proposition 4.5 *Let*

$$K_0 = i_{e_0}e^0 \wedge K_N(t)e^{-t\mathcal{R}} + e^0 \wedge i_{e_0}K_D(t)e^{-t\mathcal{R}}. \qquad (4.46)$$

Then

$$(\partial_t + \Delta + \mathcal{R})K_0 = \mathcal{R}_0[K_N(t) - K_D(t)]e^{-t\mathcal{R}} \equiv K_1. \qquad (4.47)$$

And therefore, the solution of the model problem is

$$K = K_0 + K_0 * K_1. \qquad (4.48)$$

Proof. The first equation follows immediately from Lemma 4.4. Then we note that K_0 satisfies the absolute boundary condition and we apply the Duhamel's principle. The infinite sum terminates at the second term since $\mathcal{R}_0$ contains the nilpotent element $e^0 \wedge$. ■

4.4   Proof of the Theorem

Finally we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1: As we mentioned before, we use the metric deformation (2.16) in the variation formula (2.13). To compute the right hand side of (2.13) we rewrite it as

$$\lim_{t \to 0} \text{Tr}_s F, \ F = *_l^{-1}\frac{\partial *_l}{\partial l}e^{-t\Delta_l}. \qquad (4.49)$$

Since

$$*_l^{-1}\frac{\partial *_l}{\partial l} = -\sum_{1 \leq i,j \leq n}\frac{1}{2}\langle(g_l^{-1}\frac{\partial g_l}{\partial l})(e_i), e_j\rangle c(e_i)\hat{c}(e_j), \qquad (4.50)$$

$$= -h_{ab}c(e_a)\hat{c}(e_b)x + O(x^2). \qquad (4.51)$$

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(see [BZ, Proposition 4.15]), from Theorem 4.3 (and Theorem 3.3), we have

$$F \in \Psi_{h0,G}^{1,1,0,0}(M; E). \tag{4.52}$$

Therefore, according to Lemma 4.2,

$$\mathrm{Tr}_s(F) = t^{-\frac{1}{2}}(r(t) + t^{\frac{1}{2}} s(t)) |dt|^{\frac{1}{2}},$$

with $r(0)$ and $s(0)$ given by (4.42).

Thus we must compute the two terms in (4.42). The first term computes the contribution from the interior of the manifolds and, just as in the case of the closed manifolds, can be seen to be zero. The term that really contributes to the variation formula is the second term which computes the boundary contribution. That is where the model problem comes in.

To compute the contribution of the model problem, we need to evaluate K on the spatial diagonal and at $t = 1$, and compute the corresponding Berezin integral, then integrate finally over $\mathbb{R}_+ \times \partial M$. Restricted to the spatial diagonal and $t = 1$, we have

$$K = \frac{1}{(4\pi)^{n/2}} e^{-\mathcal{R}} + \frac{1}{(4\pi)^{n/2}} e^{-x^2} c(e_0) \hat{c}(e_0) e^{-\mathcal{R}} + \frac{1}{(4\pi)^{(n-1)/2}} f(x) \mathcal{R}_0 e^{-\mathcal{R}},$$

where

$$f(x) = -\int_0^1 \int_0^\infty (e^{-(x-x')^2/4s} + e^{-(x+x')^2/4s}) 2e^{-(x+x')^2/4(1-s)} \, dx' \, ds. \tag{4.53}$$

Thus, we obtain

$$\begin{aligned}
s(0) &= -\mathrm{rank}\,\rho \int_{\mathbb{R}_+ \times \partial M} \int^B h_{ab} e^a \wedge \hat{e}^b x \frac{1}{(4\pi)^{n/2}} e^{-x^2} e^0 \wedge \hat{e}^0 \wedge e^{-\mathcal{R}} \\
&\quad -\mathrm{rank}\,\rho \int_{\mathbb{R}_+ \times \partial M} \int^B h_{ab} e^a \wedge \hat{e}^b x \frac{1}{(4\pi)^{(n-1)/2}} f(x) \mathcal{R}_0 e^{-\mathcal{R}}.
\end{aligned}$$

The first Berezin integral here can be explicitly evaluated:

$$\begin{aligned}
&-\int_{\mathbb{R}_+ \times \partial M} \int^B h_{ab} e^a \wedge \hat{e}^b x \frac{1}{(4\pi)^{n/2}} e^{-x^2} e^0 \wedge \hat{e}^0 \wedge e^{-\mathcal{R}} \\
&= \int_{\mathbb{R}_+ \times \partial M} \frac{\partial}{\partial l} i^*(\tilde{e}(g_0, g_l)) e^{-x^2} x dx \\
&= \frac{\partial}{\partial l} \int_{\partial M} i^*(\tilde{e}(g_0, g_l)).
\end{aligned}$$

Taking t to 0 and then integrating l from 0 to 1 finally gives us the desired result. ∎

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